Dynamics and Wrinkling of Radially Propagating Fronts Inferred from Scaling Laws in Channel Geometries

Barak Galanti, Oleg Kupervasser, Zeev Olami and Itamar Procaccia*
Department of Chemical Physics,
The Weizmann Institute of Science, Rehovot 76100, Israel
**Address for fall 1997: The Rockefeller University, 1230 York Ave., New York, NY 10021*

arXiv:nlin.PS/0302022 v1 11 Feb 2003

Flame Propagation is used as a prototypical example of expanding fronts that wrinkle without limit in radial geometries but reach a simple shape in channel geometry. We show that the relevant scaling laws that govern the radial growth can be inferred once the simpler channel geometry is understood in detail. In radial geometries (in contrast to channel geometries) the effect of external noise is crucial in accelerating and wrinkling the fronts. Nevertheless, once the interrelations between system size, velocity of propagation and noise level are understood in channel geometry, the scaling laws for radial growth follow.

PACS numbers 47.27.Gs, 47.27.Jv, 05.40.+j

The main idea of this Letter is that in order to derive scaling laws for unstable front propagation in radial geometry, it is useful to study noisy propagation in channel geometries, in which the noiseless dynamics results usually in simple shapes of the advancing fronts [1]. Famous examples of such situations are Laplacian growth patterns and unstable flame fronts. Laplacian growth in a channel geometry results in a single finger whose width is uniquely determined by the channel width L, the velocity of the tip v and the surface tension [2]. Laplacian growth in radial geometries results in convoluted and complex structures whose full characterization still eludes repeated theoretical attacks [2–5]. Flame propagation offers a similar situation; in channel geometries the flame front looks like a giant cusp whose velocity v is constant [6]. In radial geometries the flame front accelerates with time, and while propagating it wrinkles by the addition of hierarchies of cusps of all sizes [7–12].

The understanding of radial geometries requires control of the effect of noise on the unstable dynamics of propagation. It is particularly difficult to achieve such a control in radial geometries due to the vagueness of the distinction between external noise and noisy initial conditions. Channel geometries are simpler when they exhibit a stable solution for growth in the noiseless limit. One can then study the effects of external noise in such geometries without any ambiguity. If one finds rules to translate the resulting understanding of the effects of noise in channel growth to radial geometries, one can derive the scaling laws in the later situation in a satisfactory manner. We will exemplify the details of such a translation in the context of premixed flames that exist as self sustaining fronts of exothermic chemical reactions in gaseous combustion. But our contention is that similar ideas should be fruitful also in other contexts of unstable front propagation. Needless to say, there are aspects of the front dynamics and statistics in the radial geometry that *cannot* be explained from observations of fronts in a channel geometry; examples of such aspects are discussed at the end of this Letter.

Mathematically our example is described [8] by an equation of motion for the angle- dependent modulus of the radius vector of the flame front, $R(\theta,t)$:

$$\frac{\partial R}{\partial t}=\frac{U_b}{2{R_0}^2(t)}\left(\frac{\partial R}{\partial \theta}\right)^2+\frac{D_M}{{R_0}^2(t)}\frac{\partial^2 R}{\partial \theta^2}+\frac{\gamma U_b}{2R_0(t)}I(R)+U_b\ . \tag{1}$$

Here $0<\theta<2\pi$ is an angle and the constants U_b, D_M and γ are the front velocity for an ideal cylindrical front, the Markstein diffusivity and the thermal expansion coefficient respectively. $R_0(t)$ is the mean radius of the propagating flame:

$$R_0(t)=\frac{1}{2\pi}\int_0^{2\pi}R(\theta,t)d\theta\ . \tag{2}$$

The functional $I(R)$ is best represented in terms of its Fourier decomposition. Its Fourier component is $|k|R_k$ where R_k is the Fourier component of R. Simulations of this equation, as well as experiments in the parameter regime for which this equation is purportedly relevant, indicate that for large times R_0 grows as a power in time

$$R_0(t)=(const+t)^\beta\ , \tag{3}$$

with $\beta>1$, and that the width of the interface W grows with R_0 as

$$W(t)\sim R_0(t)^\chi\ , \tag{4}$$

with $\chi<1$. Note that this exponent may differ significantly from the exponent which characterizes the scaling of correlation functions [13]. This phenomenon is common to cases in which large features dominate the width of the graph. In a previous publication [11] we showed that β and χ satisfy the scaling relation $\beta = 1/\chi$. In this Letter we determine the numerical value of these exponents from the analysis of noisy channel growth.

In channels there is a natural lengthscale, the width $\tilde{L}$ of the channel. The translation of channel results to radial geometry will be based on the identification in the latter context of the time dependent scale $\mathcal{L}(t)$ that plays the role of $\tilde{L}$ in the former. To do this we need first to review briefly the main pertinent results for noisy channel growth. In channel geometry the equation of motion is written in terms of the position $h(x,t)$ of the flame front above the x-axis. After appropriate rescalings [6] it reads:

$$\frac{\partial h(x,t)}{\partial t} = \frac{1}{2}\left[\frac{\partial h(x,t)}{\partial x}\right]^2 + \nu \frac{\partial^2 h(x,t)}{\partial x^2} + I\{h(x,t)\} + 1. \tag{5}$$

It is convenient to rescale the domain size further to $0 < \theta < 2\pi$, and to change variables to $u(\theta,t) \equiv \partial h(\theta,t)/\partial\theta$. In terms of this function we find

$$\frac{\partial u(\theta,t)}{\partial t} = \frac{u(\theta,t)}{L^2}\frac{\partial u(\theta,t)}{\partial \theta} + \frac{\nu}{L^2}\frac{\partial^2 u(\theta,t)}{\partial \theta^2} + \frac{1}{L} I\{u(\theta,t)\} \tag{6}$$

where $L = \tilde{L}/2\pi$. In noiseless conditions this equation admits exact solutions that are represented in terms of N poles whose position $z_j(t) \equiv x_j(t) + iy_j(t)$ in the complex plane is time dependent:

$$u(\theta,t) = \nu \sum_{j=1}^{N} \cot\left[\frac{\theta - z_j(t)}{2}\right] + c.c. \ , \tag{7}$$

The steady state for channel propagation is unique and linearly stable; it consists of $N(L)$ poles which are aligned on one line parallel to the imaginary axis. The geometric appearance of the flame front is a giant cusp, analogous to the single finger in the case of Laplacian growth in a channel. The height of the cusp is proportional to L, and the propagation velocity is a constant of the motion. The number of poles in the giant cusp is linear in L,

$$N(L) = \left[\frac{1}{2}\left(\frac{L}{2\pi} + 1\right)\right] \ . \tag{8}$$

The introduction of additive random noise to the dynamics changes the picture qualitatively. It is convenient to add noise to the equation of motion in Fourier representation by adding a white noise η_k for every k mode. The noise correlation function satisfies the relation $< \eta_k(t)\eta_{k'}(t') >= \delta(k+k')\delta(t-t')2f/L$. The noise in our simulations is taken from a flat distribution in the interval $[-\sqrt{2f/L}, \sqrt{2f/L}]$; this guarantees that when the system size changes, the typical noise per unit length of the flame front remains constant. It was shown [12] that for moderate but fixed noise levels the average velocity v of the front increases with L as a power law. In our present simulations we found

$$v \sim L^{\mu} \ , \quad \mu \approx 0.35 \pm 0.03 \ . \tag{9}$$

For a fixed system size L the velocity has also a power law dependence on the level of the noise, but with a much smaller exponent: $v \sim f^{\xi}$, $\xi \approx 0.02$. These results were understood theoretically by analysing the noisy creation of new poles that interact with the poles defining the giant cusp [12]. It was shown that the system is always linearly stable or marginally stable for the introduction of new poles, but with increasing L all the eigenvalues (which are negative for a positive measure of L values) all decrease in absolute magnitude like $1/L^2$, making the system more and more susceptible to noisy perturbations [12]. The picture used remains valid as long as the poles that are introduced by the noisy perturbation do not destroy the identity of the giant cusp. Indeed, the numerical simulations show that in the presence of moderate noise the additional poles appear as smaller cusps that are constantly running towards the giant cusp. Our point here, is not to predict the numerical values of the scaling exponents in the *channel* (this was done in ref. [12], but to use them to predict the scaling exponents characterizing the acceleration and the geometry of the flame front in *radial* geometry.

Superficially it seems that in radial geometry the growth pattern is qualitatively different. In fact, close observation of the growth patterns (see Fig.1) shows that most of the time there exist some big cusps that attract other smaller cusps, but that every now and then “new” big cusps form and begin to act as local absorbers of small cusps that appear randomly. The understanding of this phenomenon gives the clue how to translate results from channels to radial growth.

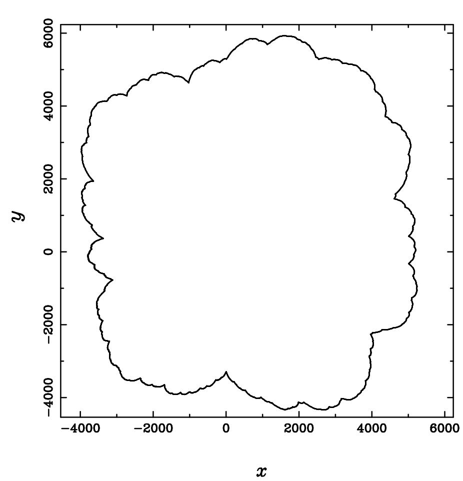

FIG. 1. Simulations of the outward propagating flame front. Note that there is a wide distribution of cusp sizes.

In the radial geometry we non-dimensionalize Equation (2) by rescaling $R \to r, R_0 \to r_0$ and $t \to \tau$, and finally rewriting the equation in terms of $u \equiv \partial r/\partial\theta$:

$$\frac{\partial u}{\partial \tau} = \frac{u}{r_0^2}\frac{\partial u}{\partial \theta} + \frac{1}{r_0^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\gamma}{2r_0} I\{u\} \ . \tag{10}$$

To complete this equation we need a second one for $r_0(t)$. This equation is obtained by averaging (2) over the angles (i.e. operating on the equation with $\frac{1}{2\pi}\int_0^{2\pi} d\theta$):

$$\frac{dr_0}{d\tau} = \frac{1}{2r_0^2}\frac{1}{2\pi}\int_0^{2\pi} u^2 d\theta + 1 \ . \tag{11}$$

These equations again admit exact solutions in terms of poles, of the form of Eq.(7). It is easy write down the equations of motion of the poles and check that the poles are attractive along the real direction (which means physically that they are attracted along the angular coordinate) but they are repulsive along the imaginary direction, which is associated with the radial coordinate. If it were not for the stretching that is caused by the increase of the radius (and with it the perimeter), all the poles would have coalesced into one giant cusp. Thus we have a competition between pole attraction and stretching. Since the attraction decreases with the distance between the poles in the angular directions, there is always an initial critical length scale above which poles cannot coalesce their real coordinates when time progresses.

Suppose now that noise adds new poles to the system. The poles do not necessarily merge their real positions with existing cusps. If we have a large cusp made from the merging of the real coordinates x_c of N_c poles, we want to know whether a nearby pole with real coordinate x_1 will merge with this large cusp. The answer will depend of course on the distance $D \equiv r_0|x_c - x_1|$. A direct calculation [11], using the equation of motions for the poles shows that there exists a critical length $\mathcal{L}(r_0)$ such that if $D > \mathcal{L}(r_0)$ the single pole never merges with the giant cusp. The result of the calculation is that

$$\mathcal{L} \sim r_0^{1/\beta} \ . \tag{12}$$

Note that a failure of a single pole to be attracted to a large cusp means that tip-splitting has occurred. This is the exact analog of tip-splitting in Laplacian growth.

It is now time to relate the channel and radial geometries. We identify the typical scale in the radial geometry as $\mathcal{L} \sim W \sim r_0^{\chi}$. On the one hand this leads to the scaling relation $\beta = 1/\chi$. On the other hand we use the result established in a channel, (9), with this identification of a scale, and find $\dot r_0 = r_0^{\chi\mu}$. Comparing with (3) we find:

$$\beta = \frac{1}{(1-\chi\mu)} \ . \tag{13}$$

This result leads us to expect two dynamical regimes for our problem. Starting from smooth initial conditions, in relatively short times the roughness exponent remains close to unity. This is mainly since the typical scale $\mathcal{L}$ is not relevant yet, and most of the poles that are generated by noise merge into a few larger cusps. In later times the roughening exponent settles at its asymptotic value, and all the asymptotic scaling relations used above become valid. We thus expect β to decrease from $1/(1-\mu)$ to an asymptotic value determined by $\chi = 1/\beta$ in (13):

$$\beta = 1 + \mu \approx 1.35 \pm 0.03 \ . \tag{14}$$

The expected value of χ is thus $\chi = 0.74 \pm 0.03$.

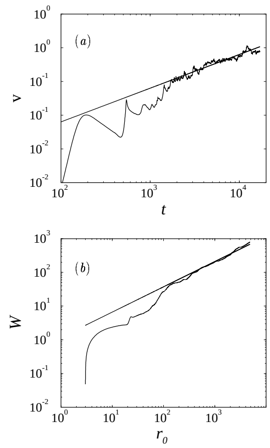

FIG. 2. Panel a: a logarithmic plot of the velocity versus time for a radially evolving system. The parameters of the simulation are: $f = 10^{-8}$, $\gamma = 0.8$, $\nu = 1$. Panel b:Logarithmic plot of the width of the flame front as a function of the mean radius.

We tested these predictions in numerical simulations. We integrated Eq.(6), and in Fig. 2 we display the results for the growth velocity as a function of time. After a limited domain of exponential growth we observe a continuous reduction of the time dependent exponent. In the initial region we get $\beta = 1.65 \pm 0.1$ while in the final decade of the temporal range we find $\beta = 1.35 \pm 0.1$. We consider this a good agreement with (14). A second important test is provided by measuring the width of the system as a function of the radius, see Fig.2b. Again we observe a cross-over related to the initial dynamics; In the last temporal decade the exponent settles at $\chi = 0.75 \pm .1$. We conclude that at times large enough to observe the asymptotics our predictions are verified.

Finally, we stress some differences between radial and channel geometries. Fronts in a channel exhibit mainly one giant cusp which is only marginally disturbed by the small cusps that are introduced by noise. In the radial geometry, as can be concluded from the discussion above, there exist at any time cusps of all sizes from the smallest to the largest. This broad distribution of cusps (and scales) must influence correlation function in ways that differ qualitatively from correlation functions computed in channel geometries.

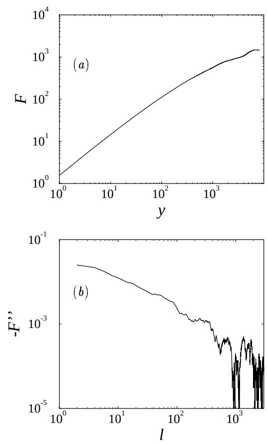

FIG. 3. Panel a: a logarithmic plot of the correlation function of the interface. Panel b: Second derivative of the correlation function of the interface.

To make the point clear we exhibit in Fig.3a the structure function

$$F(y) \equiv \sqrt{\langle |R(x+y) - R(x)|^2 \rangle} \tag{15}$$

computed for a typical radial front, with $x = R\theta$. To stress the scaling region we exhibit the second derivative of this function in Fig.3b. The low end of the graph can be fitted well by a power law $y^{-\alpha}$ with $\alpha \approx 0.6$. This indicates that $F(y) \approx Ay + By^{2-\alpha}$. In a channel geometry we get entirely different structure functions that do not exhibit such scaling functions at all. The way to understand this behavior in the radial geometry is to consider a distribution of cusps that remain distinct from each other but whose scales are distributed according to some distribution $P(\ell)$. For each of these cusps there is a contribution to the correlation function of the form $f(y,\ell) \approx \ell g(y/\ell)$ where $g(x)$ is a scaling function, $g(x) \approx x$ for $x < 1$ and $g(x) \approx$ constant for $x > 1$. The total correlation can be estimated (when the poles are distinct) as

$$F(y) \approx \sum_\ell P(\ell) \ell g(y/\ell) \ . \tag{16}$$

The first derivative leaves us with $\sum_\ell P(\ell) g'(y/\ell)$, and using the fact that g' vanishes for $x > 1$ we estimate $F'(y) = \sum_{\ell=y}^W P(\ell)$. The second derivative yields $F''(y) \approx -P(y)$. Thus the structure function is determined by the scale distribution of cusps, and if the latter is a power law, this should be seen in the second derivative of $F(y)$ as demonstrated in Fig.3. The conclusion of this analysis is that the radial case exhibits a scaling function that characterizes the distributions of cusps, $P(\ell) \approx \ell^{-\alpha}$.

In summary, we demonstrated that it is possible to use information about noisy channel dynamics to predict nontrivial features of the radial evolution, such as the acceleration and roughening exponents. It would be worthwhile to examine similar ideas in the context of Laplacian growth patterns.

ACKNOWLEDGMENTS

This work has been supported in part by the Basic Research Fund administered by the Israel Academy for Science and Humanities. IP thanks Mitchell Feigenbaum for the hospitality at the Rockefeller University.

[1] P. Pelce, *Dynamics of Curved Fronts*, (Academic press, Boston (1988))

[2] D.A. Kessler, J. Koplik and H. Levine ,Adv. Phys. **37**, 255-339 (1986)

[3] A.-L. Barbási and H.E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, 1995).

[4] T. Viscek *Fractal Growth Phenomena* (World Scientific, Singapore, 1992)

[5] T. Halpin-Healey and Y.C. Zhang, Phys. Rep. **254**, 215 (1995).

[6] O. Thual, U.Frisch and M. Henon, J. Physique, **46**, 1485 (1985).

[7] Yu.A. Gostintsev, A.G. Istratov and Yu.V. Shulenin, Combust. Expl. Shock Waves **24**, 70 (1989)

[8] L. Filyand, G.I. Sivahshinsky and M.L. Frankel, Physica **D 72**, 110 (1994)

[9] G. Joulin, J. Phys. France, **50**, 1069 (1989).

[10] G. Joulin, Phys.Rev. **E50**, 2030 (1994)

[11] O. Kupervasser ,Z. Olami and I. Procaccia, Phys.Rev.Lett **76** ,146 (1996).

[12] Z. Olami, B. Galanti, O. Kupervasser and I. Procaccia , Phys. Rev. E (1997)

[13] Z. Olami, I. Procaccia and R. Zeitak, Phys. Rev. E, **52** 3402 (1995).